MMAT5390: Mathematical Image Processing

Assignment 2 Solutions

1. (a) $H_0(t) = \mathbf{1}_{[0,1)}$, and for any $p \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{Z} \cap [0, 2^p - 1]$,

$$H_{2^p+n}(t) = 2^{\frac{p}{2}} \left(\mathbf{1}_{[\frac{n}{2^p}, \frac{n+0.5}{2^p})} - \mathbf{1}_{[\frac{n+0.5}{2^p}, \frac{n+1}{2^p})} \right).$$

The Haar transform matrix $\tilde{H} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$.

(b)

$$\begin{aligned} A_{Haar} &= \tilde{H} A \tilde{H}^T \\ &= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 6 & 8 & 0 \\ 6 & 8 & 0 & 4 \\ 8 & 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{35}{2} & \frac{3}{2} & \frac{\sqrt{2}}{2} & \sqrt{2} \\ \frac{3}{2} & -\frac{9}{2} & -\frac{5\sqrt{2}}{2} & 2\sqrt{2} \\ \frac{\sqrt{2}}{2} & -\frac{5\sqrt{2}}{2} & 0 & -5 \\ \sqrt{2} & 2\sqrt{2} & -5 & -3 \end{pmatrix}. \end{aligned}$$

(c) $\tilde{A}_{\text{Haar}} = \begin{pmatrix} \frac{35}{2} & \frac{3}{2} & 0 & \sqrt{2} \\ \frac{3}{2} & -\frac{9}{2} & -\frac{5\sqrt{2}}{2} & 2\sqrt{2} \\ 0 & -\frac{5\sqrt{2}}{2} & 0 & -5 \\ \sqrt{2} & 2\sqrt{2} & -5 & -3 \end{pmatrix}$. Hence,

$$\begin{aligned} \tilde{A} &= \tilde{H}^T \tilde{A}_{\text{Haar}} \tilde{H} \\ &= \frac{1}{4} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{35}{2} & \frac{3}{2} & 0 & \sqrt{2} \\ \frac{3}{2} & -\frac{9}{2} & -\frac{5\sqrt{2}}{2} & 2\sqrt{2} \\ 0 & -\frac{5\sqrt{2}}{2} & 0 & -5 \\ \sqrt{2} & 2\sqrt{2} & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} & 4 & \frac{23}{4} & \frac{31}{4} \\ 4 & \frac{13}{2} & \frac{33}{4} & \frac{1}{4} \\ \frac{23}{4} & \frac{33}{4} & 0 & 4 \\ \frac{31}{4} & \frac{1}{4} & 4 & 2 \end{pmatrix} \end{aligned}$$

2. (a) The 2D discrete Fourier transform (DFT) of an $M \times N$ image $g = (g(k,l))_{k,l}$, where $k = 0, 1, \cdots, M-1$ and $l = 0, 1, \cdots, N-1$, is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l) e^{-2\pi j (\frac{km}{M} + \frac{ln}{N})}$$

The Fourier transform matrix $U = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$.

(b)

$$
\begin{aligned}
B_{DFT} &= UBU \\
&= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{pmatrix} \\
&= \frac{1}{16}\begin{pmatrix}6 & -1-j & 0 & -1+j\\ -1-j & 4j & -1+j & 2\\ 0 & -1+j & -2 & -1-j\\ -1+j & 2 & -1-j & -4j\end{pmatrix}.
\end{aligned}
$$

(c) $\tilde{B}_{\mathrm{DFT}} = \frac{1}{16}\begin{pmatrix}6 & 0 & 0 & 0\\ 0 & 4j & 0 & 2\\ 0 & 0 & -2 & 0\\ 0 & 2 & 0 & -4j\end{pmatrix}.$

$$
\begin{aligned}
\tilde{B} &= (4U^*)\tilde{B}_{\mathrm{DFT}}(4U^*) \\
&= \frac{1}{16}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix}\begin{pmatrix}6 & 0 & 0 & 0\\ 0 & 4j & 0 & 2\\ 0 & 0 & -2 & 0\\ 0 & 2 & 0 & -4j\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{2} & 0 & 0 & 1\\ 0 & \frac{1}{2} & 1 & 0\\ 0 & 1 & \frac{1}{2} & 0\\ 1 & 0 & 0 & \frac{1}{2}\end{pmatrix}
\end{aligned}
$$

3. (a) The DFT of g is

$$
\widehat{g} = UgU = \frac{1}{16}\begin{pmatrix}9 & 5j & -1 & -5j\\ 9 & 5j & -1 & -5j\\ 9 & 5j & -1 & -5j\\ 9 & 5j & -1 & -5j\end{pmatrix}
$$

(b) Note that $\widehat{f * g} = 16\widehat{f} \otimes \widehat{g}$, where $\otimes$ denotes entrywise matrix multiplication. Thus, it is obvious that

$$
\widehat{f} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}
$$

Then, we have

$$
\begin{aligned}
f &= (4U^*)\widehat{f}(4U^*) \\
&= \begin{pmatrix}1 & -1 & 1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & 1 & -1\end{pmatrix}
\end{aligned}
$$

4. $\tilde{g}(k,l) = g(-1-k,-1-l)$ for any $-N \le k,l \le -1$. Then

$$\begin{aligned}
DFT(\tilde{g})(m,n) &= \frac{1}{N^2}\sum_{k=-N}^{-1}\sum_{l=-N}^{-1}\tilde{g}(k,l)e^{-2\pi j\frac{mk+nl}{N}} \\
&= \frac{1}{N^2}\sum_{k=-N}^{-1}\sum_{l=-N}^{-1}g(-1-k,-1-l)e^{-2\pi j\frac{mk+nl}{N}} \\
&= \frac{1}{N^2}\sum_{k'=0}^{N-1}\sum_{l'=0}^{N-1}g(k',l')e^{-2\pi j\frac{m(-1-k')+n(-1-l')}{N}} \\
&= \frac{1}{N^2}e^{2\pi j\frac{m+n}{N}}\sum_{k'=0}^{N-1}\sum_{l'=0}^{N-1}g(k',l')e^{-2\pi j\frac{-mk'-nl'}{N}} \\
&= e^{2\pi j\frac{m+n}{N}}\hat{g}(-m,-n).
\end{aligned}$$

5. (a) $\int_{\mathbb{R}}[H_0(t)]^2\,dt = \int_0^1 dt = 1.$

   For any $p \in \mathbb{N}\setminus\{0\}$ and $n \in \mathbb{Z}\cap[0,2^p-1]$,

$$\begin{aligned}
\int_{\mathbb{R}}[H_{2^p+n}(t)]^2\,dt &= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}}(2^{\frac{p}{2}})^2\,dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}}(-2^{\frac{p}{2}})^2\,dt \\
&= 2\cdot\frac{1}{2^{p+1}}\cdot 2^p = 1.
\end{aligned}$$

   (b) i. Let $m \in \mathbb{N}\setminus\{0\}$. There exists $p \in \mathbb{N}$ and $n \in \mathbb{Z}\cap[0,2^p-1]$ such that $m = 2^p+n$. Then

$$\begin{aligned}
\langle H_0, H_m\rangle &= \int_{\mathbb{R}} H_0(t)H_{2^p+n}(t)\,dt \\
&= \int_{\frac{n}{2^p}}^{\frac{n+0.5}{2^p}} 2^{\frac{p}{2}}\,dt + \int_{\frac{n+0.5}{2^p}}^{\frac{n+1}{2^p}}(-2^{\frac{p}{2}})\,dt \\
&= \frac{1}{2^{p+1}}\cdot 2^{\frac{p}{2}} + \frac{1}{2^{p+1}}\cdot(-2^{\frac{p}{2}}) = 0.
\end{aligned}$$

   ii. A. Suppose $p_1 = p_2$. Then

$$\begin{aligned}
\langle H_{m_1}, H_{m_2}\rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t)H_{2^{p_1}+n_2}(t)\,dt \\
&= \int_{\frac{n_1}{2^{p_1}}}^{\frac{n_1+0.5}{2^{p_1}}} 2^{\frac{p_1}{2}}\cdot 0\,dt + \int_{\frac{n_1+0.5}{2^{p_1}}}^{\frac{n_1+1}{2^{p_1}}}(-2^{\frac{p_1}{2}})\cdot 0\,dt \\
&\quad + \int_{\frac{n_2}{2^{p_1}}}^{\frac{n_2+0.5}{2^{p_1}}} 0\cdot 2^{\frac{p_1}{2}} + \int_{\frac{n_2+0.5}{2^{p_1}}}^{\frac{n_2+1}{2^{p_1}}} 0\cdot(-2^{\frac{p_1}{2}})\,dt = 0.
\end{aligned}$$

   B. Suppose $p_1 < p_2$, $p_1, p_2 \in \mathbb{N}$. Then $p_1 + 1 \le p_2$, $p_2 - p_1 \ge 1$. The length of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$ is equal to $\frac{1}{2^{p_2}}$

   - The length of $\left[0, \frac{n_1}{2^{p_1}}\right)$ is $\frac{n_1}{2^{p_1}}$. Since $\frac{n_1}{2^{p_1}}/\frac{1}{2^{p_2}} = n_1 2^{p_2-p_1}$, the length of $\left[0, \frac{n_1}{2^{p_1}}\right)$ is a multiple of that of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$.
   - The length of $\left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$ is $\frac{1}{2^{p_1+1}}$. Since $\frac{1}{2^{p_1+1}}/\frac{1}{2^{p_2}} = 2^{p_2-p_1-1}$, the length of $\left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$ is a multiple of that of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$.
   - The length of $\left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$ is $\frac{1}{2^{p_1+1}}$. Since $\frac{1}{2^{p_1+1}}/\frac{1}{2^{p_2}} = 2^{p_2-p_1-1}$, the length of $\left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$ is a multiple of that of $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right)$.

- Considering the possible subset relations between the supports of H_{m_1} and H_{m_2}, we notice that
  - $2^{p_2-p_1}n_1 \leq n_2 < 2^{p_2-p_1}(n_1+0.5)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1}{2^{p_1}}, \frac{n_1+0.5}{2^{p_1}}\right)$;
    or
  - $2^{p_2-p_1}(n_1+0.5) \leq n_2 < 2^{p_2-p_1}(n_1+1)$ and thus $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \subseteq \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right)$;
    or
  - $\left[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}}\right) \cap \left[\frac{n_1+0.5}{2^{p_1}}, \frac{n_1+1}{2^{p_1}}\right) = \varnothing$.

  In any case, H_{m_1} is constant on $[\frac{n_2}{2^{p_2}}, \frac{n_2+1}{2^{p_2}})$, and thus denoting the constant by c,

$$\begin{aligned}
\langle H_{m_1}, H_{m_2}\rangle &= \int_{\mathbb{R}} H_{2^{p_1}+n_1}(t) H_{2^{p_2}+n_2}(t)\, dt \\
&= c\int_{\frac{n_2}{2^{p_2}}}^{\frac{n_2+0.5}{2^{p_2}}} 2^{\frac{p_2}{2}}\, dt + c\int_{\frac{n_2+0.5}{2^{p_2}}}^{\frac{n_2+1}{2^{p_2}}} (-2^{\frac{p_2}{2}})\, dt \\
&= c[\frac{1}{2^{p_2+1}} \cdot 2^{\frac{p_2}{2}} + \frac{1}{2^{p_2+1}} \cdot (-2^{\frac{p_2}{2}})] = 0.
\end{aligned}$$